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Inverse scattering transformation for positons

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Abstract. The scattering problem for the one-dimensional Schrödinger operator with potential equal to the positon solution of the Korteweg–de Vries (KdV) equation is investigated. It is shown that the transition coefficient is equal to zero and different positon potentials can have the same reflection coefficient, i.e. the inverse scattering problem cannot be solved uniquely. It is observed that the reflection coefficient calculated for the positon solutions does not change with time in accordance with the inverse scattering method for KdV.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{1}$$

is one of the most well-studied nonlinear partial differential equations. Soliton solutions of this equation have been known for a long time and therefore this equation has attracted the attention of mathematicians and physicists. These solutions can be obtained using two different methods. Both methods use the relation between the KdV equation and the one-dimensional stationary Schrödinger equation. The first method uses a Darboux, or Bäcklund, transformation [9]. The second method is based on the inverse scattering transformation [3]. The second method is more general, since it gives the solution to the KdV equation with arbitrary initial data from the Faddeev class of functions v satisfying the estimate

$$\int_{-\infty}^{+\infty} (1+|x|)|v(x)|\,\mathrm{d}x < \infty. \tag{2}$$

In [10, 11] the Darboux transformation was used to obtain a new solution to the KdV equation, it was named *positon solution*:

$$u = \frac{16k_1^2\{2\sin^2 k_1(x+4k_1^2t+x_1) - k_1(x+12k_1^2t+x_2)\sin 2k_1(x+4k_1^2t+x_1)\}}{\{\sin 2k_1(x+4k_1^2t+x_1) - 2k_1(x+12k_1^2t+x_2)\}^2}. \tag{3}$$

The positon solution has the following important properties which differentiates it from the soliton solution.

- The positon solution $u(x,t)$ has a square singularity at a certain point $x_0(t)$ on the real line.
- The positon solution has slow oscillating decay at infinity

$$u(x,t) \sim_{x\to\pm\infty} \frac{c_\pm \sin(b_\pm x + \delta_\pm)}{x}. \tag{4}$$

- The positon solution is not a translational solution and changes its form with time.

The solutions of the ordinary differential equations with coefficients having singularities have been investigated intensively since the studies of Weyl [20]. The asymptotic behaviour of the positon solution coincides with the one of Wigner–von Neumann potentials providing an example of the Schrödinger operator with positive eigenvalues embedded into the continuous spectrum [14, 18]. The latter relation explains why the new solution was called positon in [10]. We point out that the positon solution is essentially described by three real parameters $x_1, x_2 \in \mathbb{R}, k_1 \in \mathbb{R}_+$. The parameters x_1 and x_2 define the phase and position of the positon, respectively. The time parameter t describes the evolution of the positon.

The same function has been obtained independently at the same time by solving the inverse scattering problem on the half-axis for rational reflection coefficients [4]. The positon potentials form a family of potentials having the same reflection coefficient and no bound state. It was proven that the spectral density for the Schrödinger operator with the positon potential on the half-axis vanishes at the point $E = k_1^2$. To give a unique solution for the inverse scattering problem the position of the zero of the spectral density (constant k_1) has to be added to the set of scattering data. The relations between these two approaches have been clarified for the half axis in [6].

The aim of this paper is to study the scattering problem for the Schrödinger operator with the positon potential on the whole axis. The formal scattering problem for the corresponding differential equation has already been constructed in [13] and later in [19]. The scattering matrix associated with the calculated solution is trivial and it was announced that the positon potential is 'supertransparent'. However, the obtained scattering solutions are not locally square integrable and therefore do not represent continuous spectrum eigenfunctions for the Schrödinger operator. In this paper we calculate such continuous spectrum eigenfunctions. We show that the positon potential has a limit point property at the singular point x_0, i.e. the singularity is so strong that the functions from the domain of the corresponding Schrödinger operator necessarily satisfy the Dirichlet boundary condition at this point. Hence we have that the total Schrödinger operator can be decomposed into the orthogonal sum of two self-adjoint operators acting in the Hilbert spaces $L_2(-\infty, x_0)$ and $L_2(x_0, +\infty)$. It follows that the scattering matrix for such a one-dimensional Schrödinger operator has trivial transition coefficients, i.e. the positon potential is totally untransparent.

It is proven that the inverse scattering problem for positon potentials cannot be solved uniquely even on the whole real line. This illustrates the difference between positon potentials and potentials from the Faddeev class. (The positon potential does not belong to the Faddeev class, since it has second-order singularity and decays slowly at infinity.)

This paper is organized as follows. First we recall some well known properties of positons. In section 3 we study the asymptotic behaviour of the positon solution and the corresponding scattering solution. We give a mathematically correct definition for the Schrödinger operator with the positon potential. The regular scattering solutions are constructed and the scattering matrix is calculated. In the last section we give an example of the model Schrödinger operator which has the same scattering matrix as the Schrödinger operator with the positon potential.

2. Darboux transformation and positon

Let us study in more detail the positon solution given by (3)

$$u(x,t) = \frac{32k_1^2(\sin T - k_1 g \cos T)\sin T}{(\sin 2T - 2k_1 g)^2} \tag{5}$$

where

$$T = k_1(x + 4k_1^2 t + x_1) \qquad g = x + 12k_1^2 t + x_2.$$

The aim of this paper is to study the scattering problem for the one-dimensional Schrödinger operator with the potential given by (5). The scattering solution, or the continuous spectrum eigenfunction for the Schrödinger operator is a generalized solution of the following stationary equation

$$-\psi'' + u(x,t)\psi = E\psi \qquad -\infty < x < \infty \tag{6}$$

where $E = k^2 > 0$ is the energy parameter. The following solution to the latter equation has been obtained in [13] using the Darboux transformation

$$\psi(k,x,t) = \left(-k^2 + \frac{4\mathrm{i}kk_1 \sin^2 T}{\sin 2T - 2k_1 g} - k_1^2 \frac{\sin 2T + 2k_1 g}{\sin 2T - 2k_1 g}\right) \mathrm{e}^{\mathrm{i}kx + 4\mathrm{i}k^3 t}. \tag{7}$$

The function ψ is a solution to the stationary Schrödinger equation but it depends on the time parameter t, since the potential $u(x,t)$ is a function of t. The solution ψ has the following asymptotic behaviour

$$\psi(k,x,t) \sim_{x\to\pm\infty} (-k^2 + k_1^2)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3 t} \tag{8}$$

as $x \to \infty$. The function

$$f(x,k,t) = (-k^2 + k_1^2)^{-1}\psi(k,x,t)\mathrm{e}^{-4\mathrm{i}k^3 t}$$

has the asymptotic behaviour $f \sim_{x\to\pm\infty} \mathrm{e}^{\mathrm{i}kx}$. This behaviour coincides with the plane wave $\mathrm{e}^{\mathrm{i}kx}$ and it was concluded in [13, 19] that the positon potential is super-reflectionless, i.e. that the corresponding scattering matrix is trivial

$$\begin{aligned} T_r(k,t) &= T_l(k,t) = 1 \\ R_r(k,t) &= R_l(k,t) = 0 \end{aligned} \tag{9}$$

where T_l, T_r, R_l, R_r denote the left and right transition and reflection coefficients. But the functions $\psi(k,x,t)$ and $f(k,x,t)$ have first- or second-order singularities located at the point $x_0 = x_0(t)$, which is the unique solution to the equation

$$\sin 2k_1(x + 4k_1^2 t + x_1) = 2k_1(x + 12k_1^2 t + x_2). \tag{10}$$

This is due to the fact that the positon potential has a second-order singularity at this point. It follows that the functions ψ and f are not locally square integrable and therefore cannot be equal to the continuous spectrum eigenfunctions of the corresponding Schrödinger operator. Regular solutions to the Schrödinger equation, i.e. solutions which are locally square integrable, will be calculated in section 4. First we are going to study the behaviour of the positon potential and the solution ψ at the singular point.

3. Singularity of the positon

Let us describe first the asymptotic behaviour of the positon solution at the singular point.

Lemma 3.1. *Let $x_0 = x_0(t)$ be the unique solution of (10) for some choice of the parameters x_1, x_2, k_1 and t, then the asymptotic behaviour of the function $u(x,t)$ as $x \to x_0$, is given by*

$$x(x,t) = \frac{\alpha}{(x - x_0)^2} + \mathrm{O}\left(\frac{1}{|x - x_0|}\right) \qquad x \to x_0 \tag{11}$$

where α is the real number determined as follows.

(1) *If* $\sin k_1(x_0 + 4k_1^2 t + x_1) \neq 0$, *then* $\alpha = 2$.
(2) *If* $\sin k_1(x_0 + 4k_1^2 t + x_1) = 0$, *then* $\alpha = 6$.

Proof. Let us consider the two possible cases separately.

(1) Let $\sin k_1(x_0+4k_1^2t+x_1)\neq 0$. The numerator and the denominator are infinitely many times differentiable functions, therefore using Taylor expansion around point x_0 we get the following representations for the square root of the denominator

$$\begin{aligned}\sin 2k_1(x+4k_1^2t+x_1)&-2k_1(x+12k_1^2t+x_2)\\ &=2k_1(\cos 2k_1(x_0+4k_1^2t+x_1)-1)(x-x_0)+\mathrm{O}(|x-x_0|^2)\qquad x\to x_0\end{aligned}\tag{12}$$

and for the numerator

$$\begin{aligned}16k_1^2(2\sin^2 k_1(x+4k_1^2t+x_1)&-k_1(x+12k_1^2t+x_2)\sin 2k_1(x+4k_1^2t+x_1))\\ &=16k_1^2(2\sin^2 k_1(x_0+4k_1^2t+x_1)-k_1(x_0+12k_1^2t+x_2)\sin 2k_1(x_0+4k_1^2t+x_1))\\ &\quad+\mathrm{O}(|x-x_0|)\qquad x\to x_0.\end{aligned}$$

Using (10) we get the following representation for the numerator

$$\begin{aligned}16k_1^2(2\sin^2 k_1(x+4k_1^2t+x_1)&-k_1(x+12k_1^2t+x_2)\sin 2k_1(x+4k_1^2t+x_1))\\ &=32k_1^2\sin^4 k_1(x_0+4k_1^2t+x_1)+\mathrm{O}(|x-x_0|)\qquad x\to x_0.\end{aligned}\tag{13}$$

Hence (12) and (13) imply (11) and that the constant α is equal to two.

(2) Let $\sin k_1(x_0+4k_1^2t+x_1)=0$. Again, using the Taylor expansion we get the following estimates for the numerator of (5)

$$\begin{aligned}16k_1^2(1-\cos 2k_1(x+4k_1^2t+x_1)&-k_1(x+12k_1^2t+x_2)\sin 2k_1(x+4k_1^2t+x_1))\\ &=\tfrac{32}{3}k_1^6(x-x_0)^4+\mathrm{O}(|x-x_0|^5)\qquad x\to x_0\end{aligned}$$

and for the square root of the denominator

$$\sin 2k_1(x+4k_1^2t+x_1)-2k_1(x+12k_1^2t+x_2)=-\tfrac{4}{3}k_1^3(x-x_0)^3+\mathrm{O}(|x-x_0|^4)\qquad x\to x_0.$$

The latter two formulae imply that (11) holds and the constant α is equal to six. □

The latter lemma shows that the positon solution can have singularities of different magnitude in contrast to what was claimed in [8]. The Schrödinger operator with the potential having second-order singularity $\frac{\alpha}{(x-x_0)^2}$, $\alpha\in\mathbb{R}$ has been investigated by Friedrichs [2], Sears [17], Kurss [7], Weinholtz [21] and Nilsson [15]. It was proven that if $\alpha\geqslant\frac{3}{4}$ then the differential equation has the limit point singularity at the point x_0, i.e. only one solution of the differential equation is locally square integrable in a neighbourhood of this point (see also [16]). Lemma 3.1 implies that the constant c is equal to six or two both greater than $\frac{3}{4}$ and the Schrödinger equation has a limit point singularity in both cases.

We can illustrate this phenomena by considering the homogeneous solutions $\psi=(x-x_0)^\beta$ of the Schrödinger equation with zero energy

$$-\psi''+\frac{\alpha}{(x-x_0)^2}\psi=0.$$

Substituting the homogeneous function $\psi=(x-x_0)^\beta$ into the equation we easily get the following quadratic equation for the parameter β

$$-\beta(\beta-1)+\alpha=0\Rightarrow\beta=\frac{1\pm\sqrt{1+4\alpha}}{2}.$$

For $\alpha=6,2$ we get $\beta=3,-2$ and $\beta=2,-1$, respectively. The solutions $(x-x_0)^{-2}$ and $(x-x_0)^{-1}$ are not locally square integrable in a neighbourhood of the singular point. We see that only one of the solutions for each $\alpha=2,6$ is locally square integrable.

This prompts us to further investigate the behaviour of ψ in a neighbourhood of x_0.

For convenience we re-write ψ (given by (7)), as follows

$$\psi(k,x,t)=\left(-k^2-k_1^2+\frac{2\mathrm{i}kk_1(1-\cos 2T)-4k_1^3g}{\sin 2T-2k_1g}\right)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3t}. \tag{14}$$

The asymptotic behaviour of the solution $\psi(k,x,t)$ at the singular point is described by the following lemma.

Lemma 3.2. *Let $\psi(k,x,t)$ be the singular solution (14) of the Schrödinger equation (6) with the positon potential.*

(1) If $\sin k_1(x_0+4k_1^2t+x_1)\neq 0$, then $\psi(k,x,t)$ has the following asymptotic behaviour at the singular point

$$\psi=\frac{a_1}{x-x_0}+\mathrm{O}(|x-x_0|)\qquad x\to x_0 \tag{15}$$

where

$$a_1=(-\mathrm{i}k+k_1\cot k_1(x_0+4k_1^2t+x_1))\mathrm{e}^{\mathrm{i}kx_0+4\mathrm{i}k^3t}. \tag{16}$$

(2) If $\sin k_1(x_0+4k_1^2t+x_1)=0$, then the singular solution $\psi(k,x,t)$ has the following asymptotic behaviour at the singular point

$$\psi=\frac{a_2}{(x-x_0)^2}+d_2+\mathrm{O}(|x-x_0|)\qquad x\to x_0 \tag{17}$$

where

$$\begin{aligned}a_2&=3\mathrm{e}^{\mathrm{i}kx_0+4\mathrm{i}k^3t}\\ d_2&=-\left(\frac{k^2}{2}+\frac{8k_1^2}{5}\right)\mathrm{e}^{\mathrm{i}kx_0+4\mathrm{i}k^3t}.\end{aligned} \tag{18}$$

Proof. Let us introduce the following notations

$$\begin{aligned}T_0&=k_1(x_0+4k_1^2t+x_1)\\ g_0&=k_1(x_0+12k_1^2t+x_2).\end{aligned}$$

We begin by considering the first case, i.e. when x_0 is a solution to (10), but

$$T_0\neq\pi n\qquad n=0,\pm1,\pm2,\ldots.$$

Let us denote by $f(x)$ and $q(x)$, respectively, the numerator and the denominator in the third term of (14) as follows

$$\psi(k,x,t)=\left(-k^2-k_1^2+\frac{f(x)}{q(x)}\right)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3t}. \tag{19}$$

These two functions are infinitely many times differentiable and using the Taylor expansion we get the following estimates for these functions

$$\begin{aligned}f(x)&=4k_1\sin T_0(\mathrm{i}k\sin T_0-\cos T_0)+4k_1^2(\mathrm{i}k\sin 2T_0-k_1)(x-x_0)^2+\mathrm{O}(|x-x_0|^2)\\ q(x)&=-4k_1\sin^2 T_0(x-x_0)-2k_1^2\sin 2T_0(x-x_0)^2+\mathrm{O}(|x-x_0|^3).\end{aligned}$$

It follows that

$$\left(-k^2-k_1^2+\frac{f(x)}{g(x)}\right)=\left(\frac{a_1'}{x-x_0}-\mathrm{i}k\cdot a_1'+\mathrm{O}(|x-x_0|)\right)\qquad x\to x_0 \tag{20}$$

where

$$a_1'=k_1\cot k_1(x_0+4k_1^2t+x_1)-\mathrm{i}k.$$

The series expansion of the exponential function gives us (15) and (16).

The second case can be investigated using similar methods. □

The latter lemma implies that the solution $\psi(k, x, t)$ of the stationary Schrödinger equation is not locally square integrable on the real axis. This solution has exactly the same singularity as the homogeneous solution to the Schrödinger equation with zero energy. We conclude that the solution has the singularity of a different power due to the different value of the constant α defining the singularity of the positon potential.

4. The regular solution

Since the solutions ψ and $\overline{\psi}$ of the Schrödinger equation (6), are linearly independent, every solution to the differential equation (6) is equal to a linear combination of these two functions. The following function is a regular solution to the Schrödinger equation with the positon potential

$$\Psi(k, x) = \overline{a_n}\psi - a_n\overline{\psi} \tag{21}$$

where a_n corresponds to cases 1 and 2 in lemma 3.2, i.e. $a_1 = (k_1 \cot k_1(x_0 + 4k_1^2 t + x_1) - \mathrm{i}k)\mathrm{e}^{\mathrm{i}kx_0+4\mathrm{i}k^3t}$ and $a_2 = 3\mathrm{e}^{\mathrm{i}kx_0+4\mathrm{i}k^3t}$, respectively.

Lemma 4.1. *The regular solution $\Psi(k, x, t)$ satisfies the Dirichlet boundary conditions at the point x_0, i.e.*

$$\Psi(k, x_0(t), t) = 0. \tag{22}$$

Proof. We start from the first case of lemma 3.2. Formula (21) implies that

$$\begin{aligned}\Psi(k, x) &= \overline{a_1}\left(\frac{a_1}{x - x_0} + \mathrm{O}(|x - x_0|)\right) - a_1\left(\frac{\overline{a_1}}{x - x_0} + \mathrm{O}(|x - x_0|)\right)\\ &= \mathrm{O}(|x - x_0|) \qquad x \to x_0\end{aligned}$$

and it follows that $\Psi(k, x(t), t)$ satisfies (22).

The second case can be investigated similarly. □

Formula (8) implies that the function $\Psi(k, x, t)$ has the following asymptotic behaviour for large x

$$\Psi(k, x, t) \sim_{x\to\pm\infty} (-k^2 + k_1^2)(\overline{a_n^*}\mathrm{e}^{\mathrm{i}kx} - a_n^*\mathrm{e}^{-\mathrm{i}kx}) \tag{23}$$

where where a_n^* are given by $a_1^* = (k_1 \cot k_1(x_0 + 4k_1^2 t + x_1) - \mathrm{i}k)\mathrm{e}^{\mathrm{i}kx_0}$ and $a_2^* = 3\mathrm{e}^{\mathrm{i}kx_0}$, respectively.

On each half-axis $x < x_0$, resp. $x > x_0$ the function $\Psi(k, x, t)$ is the unique (up to a multiplication constant) regular solution to the Schrödinger equation. Therefore, the scattering solutions F_- and F_+ are necessarily given by

$$F_-(k, x, t) := \begin{cases} c_1\Psi & x < x_0 \\ c_2\Psi & x > x_0 \end{cases} \tag{24}$$

$$F_+(k, x, t) := \begin{cases} c_3\Psi & x > x_0 \\ c_4\Psi & x < x_0 \end{cases} \tag{25}$$

where c_1, c_2, c_3 and c_4 are certain constants. These solutions should satisfy the following asymptotic conditions [1]:

$$F_-(k,x,t) \sim \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_l(k)\mathrm{e}^{-\mathrm{i}kx} & x \to -\infty \\ T_r(k)\mathrm{e}^{\mathrm{i}kx} & x \to +\infty \end{cases} \tag{26}$$

$$F_+(k,x,t) \sim \begin{cases} T_l(k)\mathrm{e}^{-\mathrm{i}kx} & x \to -\infty \\ \mathrm{e}^{-\mathrm{i}kx} + R_r(k)\mathrm{e}^{\mathrm{i}kx} & x \to +\infty. \end{cases} \tag{27}$$

It follows from (24), (25) and (23) that the constants are given by

$$\begin{aligned} c_1 &= (\overline{a_n^*}[-k^2 + k_1^2])^{-1} \\ c_3 &= (-a_n^*[-k^2 + k_1^2])^{-1} \\ c_2 &= c_4 = 0. \end{aligned} \tag{28}$$

This gives us the following asymptotic behaviour for the regular scattering solution

$$F_-(k,x) \sim \begin{cases} \mathrm{e}^{\mathrm{i}kx} + \left(-\dfrac{a_n^*}{\overline{a_n^*}}\right)\mathrm{e}^{-\mathrm{i}kx} & x \to -\infty \\ 0 & x \to \infty \end{cases} \tag{29}$$

and

$$F_+(k,x) \sim \begin{cases} 0 & x \to -\infty \\ \mathrm{e}^{-\mathrm{i}kx} + \left(-\dfrac{\overline{a_n^*}}{a_n^*}\right)\mathrm{e}^{\mathrm{i}kx} & x \to \infty. \end{cases} \tag{30}$$

The coefficients of the stationary scattering matrix can now be calculated

$$\begin{aligned} &R_l(k) = -\frac{a_n^*}{\overline{a_n^*}} \qquad R_r(k) = -\frac{\overline{a_n^*}}{a_n^*} \\ &T_r(k) = T_l(k) = 0. \end{aligned}$$

We have almost proven the following theorem.

Theorem 4.2. *The scattering matrix for the one-dimensional Schrödinger operator with the positon potential (5) is given by*

$$S(k) = \begin{pmatrix} 0 & R_l(k) \\ R_r(k) & 0 \end{pmatrix} \tag{31}$$

where

$$\begin{aligned} R_l(k) &= \left(\frac{k + \mathrm{i}h'}{k - \mathrm{i}h'}\mathrm{e}^{2\mathrm{i}kx_0}\right) \\ R_r(k) &= \left(\frac{k - \mathrm{i}h'}{k + \mathrm{i}h'}\mathrm{e}^{-2\mathrm{i}kx_0}\right) \\ h' &= k_1 \cot k_1(x_0 + 4k_1^2 t + x_1). \end{aligned} \tag{32}$$

For $\sin k_1(x_0 + 4k_1^2 t + x_1) = 0$ *we have* $h' = \infty$ *and* $R_l = -\mathrm{e}^{2\mathrm{i}kx_0}$, $R_r = \mathrm{e}^{-2\mathrm{i}kx_0}$.

Proof. We start by considering the case $\sin k_1(x_0 + 4k_1^2 t + x_1) \neq 0$. In this case we have $a_n^*|_{n=1} = (k_1 \cot k_1(x_0 + 4k_1^2 t + x_1) - \mathrm{i}k)\mathrm{e}^{\mathrm{i}kx_0}$, hence a direct calculation gives us

$$R_l(k) = -\frac{a_n^*}{\overline{a_n^*}}\bigg|_{n=1} = \frac{k + \mathrm{i}h'}{k - \mathrm{i}h'}\mathrm{e}^{2\mathrm{i}kx_0} \tag{33}$$

$$R_r(k) = -\frac{\overline{a_n^*}}{a_n^*}\bigg|_{n=1} = \frac{k - \mathrm{i}h'}{k + \mathrm{i}h'}\mathrm{e}^{-2\mathrm{i}kx_0} \tag{34}$$

where $h' = k_1 \cot k_1(x_0 + 4k_1^2 t + x_1) = k_1 \cot T_0$.

In the second case $n = 2$ we have

$$R_l(k) = -\frac{a_n^*}{a_n^*}\bigg|_{n=2} = -\mathrm{e}^{2\mathrm{i}kx_0} \tag{35}$$

$$R_r(k) = -\frac{a_n^*}{a_n^*}\bigg|_{n=2} = -\mathrm{e}^{-2\mathrm{i}kx_0}. \tag{36}$$

If we try to calculate the scattering matrix using (32) we get that the parameter h' is equal to infinity $h' = \infty$ and considering the limit of $R_l(k)$ and $R_r(k)$ when h' approaches the infinity we get exactly (35) and (36). □

One can now readily see that each half-axis, on both sides of the singular point x_0, is independent of each other. Hence we have $T(k) \equiv 0$ as opposed to $T(k) \equiv 1$, which was the claim in [12, 13, 19].

We would like to point out that the scattering matrix depends on the time parameter t, since the singular point $x_0 = x_0(t)$ is moving and the parameter $h' = h'(t)$ varies with time. The scattering matrix does not evolve, however, in accordance with the standard KdV rule:

$$S(k, t) = \mathrm{e}^{8\mathrm{i}k^3 t} S(k, 0).$$

We observe that this unusual behaviour is due to the strong singularity of the potential at the singular point x_0, as well as at infinity.

5. The Schrödinger operator with point interactions

In this section we consider the relations between the Schrödinger operator with the positon potential and the Schrödinger operator with point interactions. The latter operator, with the interaction at x_0, can be defined by certain boundary conditions at the point of interaction.

Let us consider the following operator

$$L_h = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \tag{37}$$

defined on the domain

$$\mathrm{Dom}(L_h) = \{\psi \in W_2^2(\mathbb{R}\backslash\{x_0\}) : \frac{\mathrm{d}\psi}{\mathrm{d}x}(k, x_0 \pm 0) = h\psi(k, x_0 \pm 0)\} \tag{38}$$

where h is a certain real constant. The operator L_h is self-adjoint on the domain $\mathrm{Dom}(L_h)$. This operator is equal to the orthogonal sum of the operators $L_h^\pm$ acting in the Hilbert spaces $L_2(-\infty, x_0)$ and $L_2(x_0, \infty)$:

$$L_h = L_h^- \oplus L_h^+.$$

The operators $L_h^\pm$ are defined by the same differential expression (37) on the domains

$$\begin{aligned} \mathrm{Dom}(L_h^-) &= \{\psi \in W_2^2(-\infty, x_0) : \frac{\mathrm{d}\psi}{\mathrm{d}x}(x_0 - 0) = h\psi(x_0 - 0)\} \\ \mathrm{Dom}(L_h^+) &= \{\psi \in W_2^2(x_0, \infty) : \frac{\mathrm{d}\psi}{\mathrm{d}x}(x_0 + 0) = h\psi(x_0 + 0)\}. \end{aligned} \tag{39}$$

The free Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi = E\Phi \qquad E = k^2 > 0 \tag{40}$$

has the following two solutions e^{ikx} and e^{-ikx}. Any function Φ satisfying equation (40) outside the singular point x_0 is given by

$$\Phi(k,x) = \begin{cases} \alpha_- e^{ikx} + \beta_- e^{-ikx} & x < x_0 \\ \alpha_+ e^{ikx} + \beta_+ e^{-ikx} & x > x_0 \end{cases}$$

where $\alpha_\pm$ and $\beta_\pm$ are certain constants. If we now take into account that the boundary conditions in (38), must be satisfied at the point x_0,

$$\frac{d}{dx}\Phi(k, x_0 \pm 0) = h\Phi(k, x_0 \pm 0) \tag{41}$$

we have

$$ik\alpha_\pm e^{ikx_0} - ik\beta_\pm e^{-ikx_0} = h(\alpha_\pm e^{ikx_0} + \beta_\pm e^{-ikx_0}).$$

The latter equalities imply that

$$\frac{\alpha_\pm}{\beta_\pm} = \frac{ik + h}{ik - h} e^{-2ikx_0} \tag{42}$$

hence we can see that

$$\begin{aligned} R_l(k) &= \frac{k + ih}{k - ih} e^{2ikx_0} \\ R_r(k) &= \frac{k - ih}{k + ih} e^{-2ikx_0}. \end{aligned} \tag{43}$$

The transition coefficients T_r and T_l are equal to zero and the stationary scattering matrix for the operator L_h coincides with the scattering matrix for the Schrödinger operator with the positon potential if we take $h = h'$, where h' is the real parameter defined by (32). So we see that the scattering matrix defined by the positon potential is not equal to the unit matrix but coincides with the scattering matrix for the Schrödinger operator with the point interaction at point x_0.

A similar problem has been considered in [4] where the scattering problem for the Schrödinger operator with the positon potential on the half-line has been studied. Only the case where the singularity of the positon is situated on the other half-axis has been considered. It was shown that the scattering matrix for the positon potential coincides in that case with the scattering matrix of the extended Schrödinger operator acting in a certain extended Hilbert space. The codimension of the original Hilbert space in the extended space was not trivial. In the case under consideration in this paper we were able to construct the operator with point interactions acting in the original Hilbert space and having the same scattering matrix as the operator with the positon potential.

The above analysis shows that the inverse scattering problem for the scattering matrix determined by the positon cannot be solved uniquely. The solution of such a problem includes Schrödinger operators with positon potentials having different values of k_1† and the second derivative operator with the boundary conditions given by (38) at the point x_0. Another type of solution can be called a half-positon, that is, a solution to the KdV equation that coincides with the original positon solution on one side of the singularity but which is identically zero on the other. These solutions cannot be obtained via Darboux transformation, since they are not meromorphic. This is in contrast to the positon solution which is a meromorphic function of the parameter x. An interesting problem is to study the interaction between half-positons as it has been done for positons and solitons in [11–13]. This question will be discussed in a future publication.

† Hence one should include k_1 in the set of scattering data in order to ensure the uniqueness of the solution to the scattering problem in the class of functions including positons, as it has been done in [5] for the scattering problem on the half-axis.

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